

Approximate l -state solutions of the D -dimensional Schrödinger equation for Manning-Rosen potential

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Abstract

The Schrödinger equation in D -dimensions for the Manning-Rosen potential with the centrifugal term is solved approximately to obtain bound states eigensolutions (eigenvalues and eigenfunctions). The Nikiforov-Uvarov (NU) method is used in the calculations. We present numerical calculations of energy eigenvalues to two- and four-dimensional systems for arbitrary quantum numbers n and l with three different values of the potential parameter α . It is shown that because of the interdimensional degeneracy of eigenvalues, we can also reproduce eigenvalues of a upper/lower dimensional sytem from the well-known eigenvalues of a lower/upper dimensional system by means of the transformation $(n, l, D) \rightarrow (n, l \pm 1, D \mp 2)$.. This solution reduces to the Hulthén potential case.

Keywords: Bound states; Manning-Rosen potential; Nikiforov-Uvarov method.

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I. INTRODUCTION

One of the important tasks of quantum mechanics is to find exact solutions of the wave equations (nonrelativistic and relativistic) for certain potentials of physical interest since they contain all the necessary information regarding the quantum system under consideration. It is well known that the exact solutions of these wave equations are only possible in a few simple cases such as the Coulomb, the harmonic oscillator, pseudoharmonic and Mie-type potentials [1-8]. For an arbitrary l -state, most quantum systems could be only treated by approximation methods. For the rotating Morse potential some semiclassical and/or numerical solutions have been obtained by using Pekeris approximation [9-13]. In recent years, many authors have studied the nonrelativistic and relativistic wave equations with certain potentials for the s - and l -cases. The exact and approximate solutions of these models have been obtained analytically [10-14].

Many exponential-type potentials have been solved like the Morse potential [12,13,15], the Hulthén potential [16-19], the Pöschl-Teller [20], the Woods-Saxon potential [21-23], the Kratzer-type potentials [12,14,24-27], the Rosen-Morse-type potentials [28,29], the Manning-Rosen potential [29-33] and other multiparameter exponential-type potentials [34,35]. Various methods are used to obtain the exact solutions of the wave equations for this type of exponential potentials (cf. [36] and the references therein).

Recently, the NU method [37] has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems. In this work, we attempt to apply this method to study another exponential-type potential proposed by Manning and Rosen [29-33]. With an approximation to centrifugal term, we solve the D -dimensional Schrödinger equation to its bound states energies and wavefunctions. This potential is defined as [29-33]

$$V(r) = -V_0 \frac{e^{-r/b}}{1 - e^{-r/b}} + V_1 \left(\frac{e^{-r/b}}{1 - e^{-r/b}} \right)^2, \quad V_0 = \frac{A}{\kappa b^2}, \quad V_1 = \frac{\alpha(\alpha - 1)}{\kappa b^2}, \quad \kappa = 2\mu/\hbar^2, \quad (1)$$

where A and α are two-dimensionless parameters [27,28] but the screening parameter b has dimension of length which has a potential range $1/b$. The potential (1) may be further put in the following simple form

$$V(r) = -\frac{Ce^{-r/b} + De^{-2r/b}}{(1 - e^{-r/b})^2}, \quad C = A, \quad D = -A - \alpha(\alpha-1), \quad (2)$$

which is usually used for the description of diatomic molecular vibrations [38,39]. It is also used in several branches of physics for their bound states and scattering properties. The potential in (1) remains invariant by mapping $\alpha \rightarrow 1 - \alpha$ and has a relative minimum value $V(r_0) = -\frac{A^2}{4\kappa b^2\alpha(\alpha-1)}$ at $r_0 = b \ln \left[1 + \frac{2\alpha(\alpha-1)}{A}\right]$ for $\alpha > 0$ to be obtained from the first derivative $\left.\frac{dV}{dr}\right|_{r=r_0} = 0$. The second derivative which determines the force constants at $r = r_0$ is given by

$$\left.\frac{d^2V}{dr^2}\right|_{r=r_0} = \frac{A^2 [A + 2\alpha(\alpha - 1)]^2}{8b^4\alpha^3(\alpha - 1)^3}. \quad (3)$$

The contents of this paper are as follows: In Section II we outline the Nikiforov-Uvarov (NU) method. In Section III, we derive $l \neq 0$ bound state eigensolutions of the D -dimensional Schrödinger equation for the Manning-Rosen potential by this method. In Section IV, we present our numerical calculations in $2D$ and $4D$ systems for various quantum numbers n and l . Section V, is devoted to for two special cases, namely, $l = 0$ and the Hulthén potential. The concluding remarks are given in Section VI.

II. THE NIKIFOROV-UVAROV METHOD

The NU method is based on solving the second-order linear differential equation by reducing it to a generalized equation of hypergeometric type [37]. In this method after employing an appropriate coordinate transformation $z = z(r)$, the Schrödinger equation can be written in the following form:

$$\psi_n''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi_n'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi_n(z) = 0, \quad (4)$$

$$\psi_n(z) = \phi_n(z)y_n(z),$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are the polynomials with at most of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. The special orthogonal polynomials [37] reduce Eq. (4) to a simple equation of the following hypergeometric type:

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0, \quad (5)$$

where

$$\begin{aligned} \tau(z) &= \tilde{\tau}(z) + 2\pi(z), \quad \tau'(z) < 0, \\ \sigma(z) &= \pi(z) \frac{\phi(z)}{\phi'(z)}, \end{aligned} \quad (6)$$

and λ is a constant given in the form

$$\begin{aligned} \lambda = \lambda_n &= -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z), \quad n = 0, 1, 2, \dots \\ \lambda &= k + \pi'(z). \end{aligned} \quad (7)$$

It is worthwhile to note that λ or λ_n are obtained from a particular solution of the form $y(z) = y_n(z)$ which is a polynomial of degree n . Further, $y_n(z)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (8)$$

where B_n is the normalization constant and the weight function $\rho(z)$ must satisfy the condition [37]

$$\frac{d}{dz}w(z) = \frac{\tau(z)}{\sigma(z)}w(z), \quad w(z) = \sigma(z)\rho(z). \quad (9)$$

In order to determine the weight function given in Eq. (9), we must obtain the following polynomial:

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)}. \quad (10)$$

In principle, the expression under the square root sign in Eq. (10) can be arranged as the square of a polynomial. This is possible only if its discriminant is zero. In this case, an equation for k is obtained. After solving this equation, the obtained values of k are included in the NU method and here there is a relationship between λ and k given in Eq. (7).

III. BOUND-STATE SOLUTIONS FOR ARBITRARY L -STATE

In this section, we follow closely the approach of Ref. [36]. We begin by considering the SE, in arbitrary dimension D , as [40-42]

$$\begin{aligned} & \left\{ \nabla_D^2 + \frac{2\mu}{\hbar^2} [E_{nl} - V(r)] \right\} \psi_{l_1 \dots l_{D-2}}^{(l_{D-1}=l)}(\mathbf{x}) = 0, \\ & \nabla_D^2 = \frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r} \frac{\partial}{\partial r} \\ & + \frac{1}{r^2} \left[\frac{1}{\sin^{D-2} \theta_{D-1}} \frac{\partial}{\partial \theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{\partial}{\partial \theta_{D-1}} \right) - \frac{L_{D-2}^2}{\sin^2 \theta_{D-1}} \right], \\ & \psi_{l_1 \dots l_{D-2}}^{(l)}(\mathbf{x}) = R_l(r) Y_{l_1 \dots l_{D-2}}^{(l)}(\hat{\mathbf{x}}), \quad R_l(r) = r^{-(D-1)/2} g(r), \end{aligned} \quad (11)$$

where the potential $V(r)$ is taken as the Manning-Rosen form in (1). In addition, μ and E_{nl} denote the reduced mass and energy of two interacting particles, respectively. \mathbf{x} is a D -dimensional position vector with the hyperspherical Cartesian components x_1, x_2, \dots, x_D given as follows [43-47]:

$$\begin{aligned} x_1 &= r \cos \theta_1 \sin \theta_2 \cdots \sin \theta_{D-1}, \\ x_2 &= r \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{D-1}, \\ x_3 &= r \cos \theta_2 \sin \theta_3 \cdots \sin \theta_{D-1}, \\ &\vdots \\ x_j &= r \cos \theta_{j-1} \sin \theta_j \cdots \sin \theta_{D-1}, \quad 3 \leq j \leq D-1, \\ &\vdots \\ x_{D-1} &= r \cos \theta_{D-2} \sin \theta_{D-1}, \end{aligned}$$

$$x_D = r \cos \theta_{D-1}, \quad \sum_{j=1}^D x_j^2 = r^2, \quad (12)$$

for $D = 2, 3, \dots$. We have $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$ for $D = 2$ and $x_1 = r \cos \varphi \sin \theta$, $x_2 = r \sin \varphi \sin \theta$, $x_3 = r \cos \theta$ for $D = 3$. The Laplace operator ∇_D^2 is defined by [48]

$$\nabla_D^2 = \sum_{j=1}^D \frac{\partial^2}{\partial x_j^2}. \quad (13)$$

The volume element of the configuration space is given by

$$\prod_{j=1}^D dx_j = r^{D-1} dr d\Omega, \quad d\Omega = \prod_{j=1}^{D-1} (\sin \theta_j)^{j-1} d\theta_j, \quad (14)$$

where $r \in [0, \infty)$, $\theta_1 \in [0, 2\pi]$ and $\theta_j \in [0, \pi]$, $j \in [2, D-1]$. Equation (13) permits a solution via separation of variables, if one writes generalized spherical harmonics $Y_{\ell_1 \dots \ell_{D-2}}^{(\ell)}(\hat{\mathbf{x}})$ [49]:

$$Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}) = Y(l_1, l_2, \dots, l_{D-2}, l), \quad l = |m| \text{ for } D = 2,$$

$$Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}} = \theta_1, \theta_2, \dots, \theta_{D-1}) = \prod_{j=1}^{D-1} H(\theta_j), \quad (15)$$

as a simultaneous eigenfunction of L_j^2 :

$$L_1^2 Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}) = m^2 Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}),$$

$$L_j^2 Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}) = l_j(l_j + j - 1) Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}), \quad j \in [1, D-1],$$

$$l = 0, 1, \dots, l_k = 0, 1, \dots, l_{k+1}, \quad k \in [2, D-2],$$

$$l_1 = -l_2, -l_2 + 1, \dots, l_2 - 1, l_2,$$

$$L_{D-1}^2 Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}) = l(l + D - 2) Y_{\ell_1 \dots \ell_{D-2}}^{(l)}(\hat{\mathbf{x}}). \quad (16)$$

The unit vector along \mathbf{x} is usually denoted by $\hat{\mathbf{x}} = \mathbf{x}/r$. Additionally, the angular momentum operators L_j^2 are defined as [43-47,49]:

$$L_1^2 = -\frac{\partial^2}{\partial \theta_1^2},$$

$$L_k^2 = \sum_{a < b=2}^{k+1} L_{ab}^2 = -\frac{1}{\sin^{k-1} \theta_k} \frac{\partial}{\partial \theta_k} \left(\sin^{k-1} \theta_k \frac{\partial}{\partial \theta_k} \right) + \frac{L_{k-1}^2}{\sin^2 \theta_k}, \quad 2 \leq k \leq D-1,$$

$$L_{ab} = -i \left[x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a} \right]. \quad (17)$$

The substitution of Eqs. (13) and (15)-(17) into Eq. (11) allows us to obtain, via the method of separation of variables, the following equation:

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{1}{\sin^{D-2} \theta_{D-1}} \frac{\partial}{\partial \theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{\partial}{\partial \theta_{D-1}} \right) - \frac{L_{D-2}^2}{\sin^2 \theta_{D-1}} \right. \\ \left. + \frac{2\mu}{\hbar^2} E_{nl} - \frac{\alpha(\alpha-1)}{b^2} \frac{e^{-2r/b}}{(1-e^{-r/b})^2} + \frac{A}{b^2} \frac{e^{-r/b}}{1-e^{-r/b}} - \frac{l(l+D-2)}{r^2} \right\}$$

$$\times r^{-(D-1)/2} g(r) Y_{l_1 \dots l_{D-2}}^{(l)}(\hat{\mathbf{x}}) = 0,$$

$$-L_{D-1}^2 = \frac{1}{\sin^{D-2} \theta_{D-1}} \frac{\partial}{\partial \theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{\partial}{\partial \theta_{D-1}} \right) - \frac{L_{D-2}^2}{\sin^2 \theta_{D-1}},$$

$$-L_{D-2}^2 = \frac{1}{\sin^{D-3} \theta_{D-2}} \frac{\partial}{\partial \theta_{D-2}} \left(\sin^{D-3} \theta_{D-2} \frac{\partial}{\partial \theta_{D-2}} \right) - \frac{L_{D-3}^2}{\sin^2 \theta_{D-2}},$$

⋮

$$-L_j^2 = \frac{1}{\sin^{j-1} \theta_j} \frac{\partial}{\partial \theta_j} \left(\sin^{j-1} \theta_j \frac{\partial}{\partial \theta_j} \right) - \frac{L_{j-1}^2}{\sin^2 \theta_j}, \quad j \in [2, D-2],$$

⋮

$$-L_1^2 = \frac{\partial^2}{\partial \theta_1^2}, \quad (18)$$

where L_k^2 , $k \in [1, D-1]$ are the angular operators. Thus, with the aid of Eqs. (15) and (16), the last wave equation can be easily separated into the following radial and angular parts as [50,51]:¹

$$\left\{ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2} E_{nl} - V_{eff}(r) \right\} g(r) = 0,$$

$$V_{eff}(r) = \frac{1}{b^2} \left[\frac{\alpha(\alpha-1)e^{-2r/b}}{(1-e^{-r/b})^2} - \frac{Ae^{-r/b}}{1-e^{-r/b}} \right] + \frac{(D+2l-2)^2-1}{4r^2}, \quad (19)$$

$$\left[\frac{1}{\sin^{D-2} \theta_{D-1}} \frac{d}{d\theta_{D-1}} \left(\sin^{D-2} \theta_{D-1} \frac{d}{d\theta_{D-1}} \right) + l(l+D-2) - \frac{\Lambda_{D-2}}{\sin^2 \theta_{D-1}} \right] H(\theta_{D-1}) = 0, \quad (20)$$

\vdots

$$\left[\frac{1}{\sin^{j-1} \theta_j} \frac{d}{d\theta_j} \left(\sin^{j-1} \theta_j \frac{d}{d\theta_j} \right) + \Lambda_j - \frac{\Lambda_{j-1}}{\sin^2 \theta_j} \right] H(\theta_j) = 0, \quad j \in [2, D-2], \quad (21)$$

\vdots

$$\left[\frac{d^2}{d\theta_1^2} + \Lambda_1 \right] H(\theta_1) = 0, \quad \Lambda_1 = l_1^2 = m^2, \quad (22)$$

where $\Lambda_p = l_p(l_p + p - 1)$, $p \in [1, D-1]$ are separation constants and $g(r)$ is defined in Eq. (11). The solution in (22) is periodic and must satisfy the periodic boundary condition $H(\theta_1 = \varphi) = H(\theta_1 = \varphi + 2\pi)$ from which we obtain [42]

$$H(\theta_1) = \frac{1}{\sqrt{2\pi}} \exp(\pm i l_1 \theta_1), \quad l_1 = 0, 1, 2, \dots \quad (23)$$

¹The Schrödinger equation in the presence of this potential is separable to $(D-1)$ - angular equations for the angular parameters $(\theta_1 = \phi, \theta_2, \dots, \theta_{D-1} = \theta)$ and one radial equation for the radial parameter r with calculated separation constants Λ_p where $p \in [1, D-1]$.

Further, Eqs. (20) and (21) representing the angular wave equation become

$$\begin{aligned} & \frac{d^2 H(\theta_{D-1})}{d\theta_{D-1}^2} + (D-2) \frac{\cos \theta_{D-1}}{\sin \theta_{D-1}} \frac{dH(\theta_{D-1})}{d\theta_{D-1}} \\ & + \left[l(l+D-2) - \frac{\Lambda_{D-2}}{\sin^2 \theta_{D-1}} \right] H(\theta_{D-1}) = 0, \end{aligned} \quad (24)$$

$$\frac{d^2 H(\theta_j)}{d\theta_j^2} + (j-1) \frac{\cos \theta_j}{\sin \theta_j} \frac{dH(\theta_j)}{d\theta_j} + \left(\Lambda_j - \frac{\Lambda_{j-1}}{\sin^2 \theta_j} \right) H(\theta_j) = 0, \quad (25)$$

with $j \in [2, D-2]$, $D > 3$ and Λ_p which is well-known in three-dimensional space [48].²

Hence, Eqs. (24) and (25) are to be solved in the following subsection.

A. The solutions of the D -dimensional angular equations

In order to apply NU method, we introduce a new variable $s = \cos \theta_j$. Hence, Eq. (25) is then rearranged in the form of the universal associated-Legendre differential equation

$$\frac{d^2 H(s)}{ds^2} - \frac{js}{1-s^2} \frac{dH(s)}{ds} + \frac{\Lambda_j - \Lambda_{j-1} - \Lambda_j s^2}{(1-s^2)^2} H(s) = 0, \quad (26)$$

where $j \in [2, D-2]$, $D > 3$. By comparing Eqs. (26) and (4), the corresponding polynomials are obtained

$$\tilde{\tau}(s) = -js, \quad \sigma(s) = 1 - s^2, \quad \tilde{\sigma}(s) = -\Lambda_j s^2 + \Lambda_j - \Lambda_{j-1}. \quad (27)$$

Inserting the above expressions into Eq. (10) and taking $\sigma'(s) = -2s$, one obtains the following function:

$$\pi(s) = \frac{(j-2)}{2}s \pm \sqrt{\left[\left(\frac{j-2}{2} \right)^2 + \Lambda_j - k \right] s^2 + k - \Lambda_j + \Lambda_{j-1}}. \quad (28)$$

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

² $\Lambda_{D-2} = m^2$ and $l_{D-2} = m$ for $D = 3$.

$$\pi(s) = \begin{cases} \left(\frac{j-2}{2} + \tilde{\Lambda}_{j-1}\right)s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}, \\ \left(\frac{j-2}{2} - \tilde{\Lambda}_{j-1}\right)s & \text{for } k_1 = \Lambda_j - \Lambda_{j-1}, \\ \frac{j-2}{2}s + \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\frac{j-2}{2}\right)^2, \\ \frac{j-2}{2}s - \tilde{\Lambda}_{j-1} & \text{for } k_2 = \Lambda_j + \left(\frac{j-2}{2}\right)^2, \end{cases} \quad (29)$$

where $\tilde{\Lambda}_p = l_p + (p-1)/2$, with $p = j-1, j$ and $j \in [2, D-2]$, $D > 3$. Imposing the condition $\tau'(s) < 0$ in Eq. (6), one selects the following physically valid solutions:

$$k_1 = \Lambda_j - \Lambda_{j-1} \quad \text{and} \quad \pi(s) = \left(\frac{j-2}{2} - \tilde{\Lambda}_{j-1}\right)s, \quad (30)$$

which yields from Eq. (6) that

$$\tau(s) = -2(1 + \tilde{\Lambda}_{j-1})s. \quad (31)$$

Making use from Eq. (7), the following expressions for λ are obtained as follows:

$$\lambda = \lambda_{n_j} = 2n_j(1 + \tilde{\Lambda}_{j-1}) + n_j(n_j - 1), \quad (32)$$

$$\lambda = \Lambda_j - \Lambda_{j-1} - \tilde{\Lambda}_{j-1} + \frac{j-2}{2}. \quad (33)$$

Upon comparing Eqs. (32) and (33), we obtain

$$n_j = \tilde{\Lambda}_j - \tilde{\Lambda}_{j-1} - \frac{1}{2}. \quad (34)$$

In addition, using Eqs. (6) and (8)-(9), we obtain the following useful parts of the wavefunctions:

$$\phi(s) = (1 - s^2)^{l_{j-1}/2}, \quad \rho(s) = (1 - s^2)^{\tilde{\Lambda}_{j-1}}, \quad (35)$$

where $j \in [2, D-2]$, $D > 3$. Besides, substituting the weight function $\rho(s)$ given in (35) into Eq. (8), we obtain

$$y_{n_j}(s) = A_{n_j} (1 - s^2)^{-\tilde{\Lambda}_{j-1}} \frac{d^{n_j}}{ds^{n_j}} (1 - s^2)^{n_j + \tilde{\Lambda}_{j-1}}, \quad (36)$$

where A_{n_j} is the normaliation factor. Finally the angular wavefunction is

$$H_{n_j}(\theta_j) = N_{n_j} (\sin \theta_j)^{l_{j-1}} P_{n_j}^{(\tilde{\Lambda}_{j-1}, \tilde{\Lambda}_{j-1})}(\cos \theta_j), \quad j \in [2, D-2], \quad D > 3 \quad (37)$$

with n_j given in Eq. (34) becomes

$$n_j = l_j - l_{j-1}, \quad j \in [2, D-2], \quad D > 3. \quad (38)$$

Likewise, using $s = \cos \theta_{D-1}$, we can rewrite Eq. (24) in the associated Legendre form

$$\frac{d^2 H(s)}{ds^2} - \frac{(D-1)s}{1-s^2} \frac{dH(s)}{ds} + \frac{\nu(1-s^2) - \Lambda_{D-2}}{(1-s^2)^2} H(s) = 0, \quad (39)$$

$$\nu = l(l + D - 2). \quad (40)$$

It's worth to note that, Eq. (39) has been recently solved in $3D$ by the NU method in [40,41,51]. However, our aim is to solve it in D -dimensions. Hence, comparing Eqs. (39) and (4), the corresponding polynomials are obtained

$$\tilde{\tau}(s) = -(D-1)s, \quad \sigma(s) = 1 - s^2, \quad \tilde{\sigma}(s) = -\nu s^2 + \nu - \Lambda_{D-2}. \quad (41)$$

Inserting the above expressions into Eq. (10) and taking $\sigma'(s) = -2s$, one obtains:

$$\pi(s) = \frac{(D-3)}{2}s \pm \sqrt{\left[\left(\frac{D-3}{2}\right)^2 + \nu - k\right]s^2 + k - \nu + \Lambda_{D-2}}. \quad (42)$$

Following the method, the polynomial $\pi(s)$ is found to have the following four possible values:

$$\pi(s) = \begin{cases} \left(\frac{D-3}{2} + \tilde{\Lambda}_{D-2}\right)s & \text{for } k_1 = \nu - \Lambda_{D-2}, \\ \left(\frac{D-3}{2} - \tilde{\Lambda}_{D-2}\right)s & \text{for } k_1 = \nu - \Lambda_{D-2}, \\ \frac{(D-3)}{2}s + \tilde{\Lambda}_{D-2} & \text{for } k_2 = \nu + \left(\frac{D-3}{2}\right)^2, \\ \frac{(D-3)}{2}s - \tilde{\Lambda}_{D-2} & \text{for } k_2 = \nu + \left(\frac{D-3}{2}\right)^2, \end{cases} \quad (43)$$

where $\tilde{\Lambda}_{D-2} = l_{D-2} + \frac{D-3}{2}$. Imposing the condition $\tau'(s) < 0$ in Eq. (6), one selects the following physically valid solutions:

$$k_1 = \nu - \Lambda_{D-2} \quad \text{and} \quad \pi(s) = -l_{D-2}s, \quad \nu = l(l + D - 2), \quad (44)$$

giving

$$\tau(s) = -2(1 + \tilde{\Lambda}_{D-2})s. \quad (45)$$

Making use from Eq. (7), we obtain

$$\lambda = \lambda_{n_{D-1}} = 2n_{D-1}(1 + \tilde{\Lambda}_{D-2}) + n_{D-1}(n_{D-1} - 1), \quad (46)$$

$$\lambda = l(l + D - 2) - l_{D-2}(l_{D-2} + D - 2). \quad (47)$$

We compare Eqs. (46) and (47), the angular momentum l values are obtained from

$$l = n_{D-1} + l_{D-2}, \quad (48)$$

which can be easily reduced to the well-known solution

$$l = n + m, \quad (49)$$

in $3D$ [51]. Using Eqs (6) and (8)-(9), we obtain the following useful parts of the wavefunctions:

$$\phi(s) = (1 - s^2)^{l_{D-2}/2}, \quad \rho(s) = (1 - s^2)^{\tilde{\Lambda}_{D-2}}. \quad (50)$$

Besides, the Rodrigues relation (8) gives

$$y_{n_{D-1}}(s) = B_{n_{D-1}} (1 - s^2)^{-\tilde{\Lambda}_{D-2}} \frac{d^{n_{D-1}}}{ds^{n_{D-1}}} (1 - s^2)^{n_{D-1} + \tilde{\Lambda}_{D-2}}, \quad (51)$$

where $B_{n_{D-1}}$ is the normaliation factor. Finally the angular wavefunctions are

$$H_{n_{D-1}}(\theta_{D-1}) = N_{n_{D-1}} (\sin \theta_{D-1})^{l_{D-2}} P_{n_{D-1}}^{(\tilde{\Lambda}_{D-2}, \tilde{\Lambda}_{D-2})}(\cos \theta_{D-1}), \quad (52)$$

where n_{D-1} is given by Eq. (48).

B. The solutions of the D -dimensional radial equation

Since the radial part of the D -dimensional Schrödinger equation with above Manning-Rosen effective potential has no analytical solution for $l \neq 0$ states, an approximation to the centrifugal term has to be made. The good approximation for $1/r^2$ in the centrifugal barrier is taken as [18,33]³

$$\frac{1}{r^2} \approx \frac{1}{b^2} \frac{e^{-r/b}}{(1 - e^{-r/b})^2}, \quad (53)$$

in a short potential range. To solve it by the present method, we need to recast Eq. (19) with the aid of Eq. (53), into the form of Eq. (4) changing the variables $r \rightarrow z$ through the mapping function $r = f(z)$ and energy transformation given by

$$z = e^{-r/b}, \quad \varepsilon = \sqrt{-\frac{2\mu b^2 E_{nl}}{\hbar^2}}, \quad E_{nl} < 0, \quad (54)$$

to obtain the following hypergeometric equation:

$$\begin{aligned} & \frac{d^2 g(z)}{dz^2} + \frac{(1-z)}{z(1-z)} \frac{dg(z)}{dz} \\ & + \frac{1}{[z(1-z)]^2} \left\{ -\varepsilon^2 + \frac{1}{4} [4A + 8\varepsilon^2 - (D + 2l - 2)^2 + 1] z - [A + \varepsilon^2 + \alpha(\alpha - 1)] z^2 \right\} g(z) = 0. \end{aligned} \quad (55)$$

We notice that for bound state (real) solutions, the last equation requires that

$$z = \begin{cases} 0, & \text{when } r \rightarrow \infty, \\ 1, & \text{when } r \rightarrow 0, \end{cases} \quad (56)$$

and thus the finite radial wavefunctions $R_{nl}(z) \rightarrow 0$. To apply the NU method, it is necessary to compare Eq. (15) with Eq. (4). Subsequently, the following value for the parameters in Eq. (4) are obtained as

³The series approximation to the expression $\frac{1}{b^2} \frac{e^{-2r/b}}{(1 - e^{-r/b})^2} \approx \frac{1}{r^2} - \frac{1}{br}$, it includes a Coulomb term.

$$\tilde{\tau}(z) = 1 - z, \quad \sigma(z) = z - z^2$$

$$\tilde{\sigma}(z) = - \left[A + \varepsilon^2 + \alpha(\alpha - 1) \right] z^2 + \frac{1}{4} \left[4A + 8\varepsilon^2 - (D + 2l - 2)^2 + 1 \right] z - \varepsilon^2. \quad (57)$$

If one inserts these values of parameters into Eq. (10), with $\sigma'(z) = 1 - 2z$, the following linear function is achieved

$$\pi(z) = -\frac{z}{2}$$

$$\pm \frac{1}{2} \sqrt{\{1 + 4[A + \varepsilon^2 + \alpha(\alpha - 1)] - k\} z^2 + \{4k - [4A + 8\varepsilon^2 - (D + 2l - 2)^2 + 1]\} z + 4\varepsilon^2}. \quad (58)$$

According to this method the expression in the square root has to be set equal to zero, that is, $\Delta = \{1 + 4[A + \varepsilon^2 + \alpha(\alpha - 1)] - k\} z^2 + \{4k - [4A + 8\varepsilon^2 - (D + 2l - 2)^2 + 1]\} z + 4\varepsilon^2 = 0$. Thus the constant k can be obtained as

$$k = A - \frac{(D + 2l - 2)^2 - 1}{4} \pm a\varepsilon, \quad a = \sqrt{(1 - 2\alpha)^2 + (D + 2l - 2)^2 - 1}. \quad (59)$$

In view of that, we can find four possible functions for $\pi(z)$ as

$$\pi(z) = -\frac{z}{2} \pm \begin{cases} \varepsilon - \left(\varepsilon - \frac{a}{2}\right) z, & \text{for } k = A - \frac{(D+2l-2)^2-1}{4} + a\varepsilon, \\ \varepsilon - \left(\varepsilon + \frac{a}{2}\right) z; & \text{for } k = A - \frac{(D+2l-2)^2-1}{4} - a\varepsilon. \end{cases} \quad (60)$$

We must select

$$k = A - \frac{(D + 2l - 2)^2 - 1}{4} - a\varepsilon, \quad \pi(z) = -\frac{z}{2} + \varepsilon - \left(\varepsilon + \frac{a}{2}\right) z, \quad (61)$$

in order to obtain the polynomial, $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ having negative derivative as

$$\tau(z) = 1 + 2\varepsilon - (2 + 2\varepsilon + a)z, \quad \tau'(z) = -(2 + 2\varepsilon + a). \quad (62)$$

We can also write the values of $\lambda = k + \pi'(z)$ and $\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z)$, $n = 0, 1, 2, \dots$ as

$$\lambda = A - \frac{(D + 2l - 2)^2 - 1}{4} - (1 + a) \left[\frac{1}{2} + \varepsilon \right], \quad (63)$$

$$\lambda_n = n(1 + n + a + 2\varepsilon), \quad n = 0, 1, 2, \dots \quad (64)$$

respectively. Additionally, using the definition of $\lambda = \lambda_n$ and solving the resulting equation for ε , allows one to obtain

$$\varepsilon = \frac{4(n+1)^2 + (D+2l-2)^2 - 1 + 4(2n+1)\eta - 4A}{8(n+1+\eta)}, \quad \eta = \frac{-1+a}{2}, \quad (65)$$

from which we obtain the discrete energy levels

$$E_{nl}^{(D)} = -\frac{\hbar^2}{32\mu b^2} \left[\frac{4(n+1)^2 + (D+2l-2)^2 + 4(2n+1)\eta - 4A - 1}{2(n+1+\eta)} \right]^2, \quad 0 \leq n, l < \infty \quad (66)$$

where n denotes the radial quantum number. It is found that Λ remains invariant by mapping $\alpha \rightarrow 1 - \alpha$, so do the bound state energies E_{nl} . An important quantity of interest for the Manning-Rosen potential is the critical coupling constant A_c , which is that value of A for which the binding energy of the level in question becomes zero. Using Eq. (26), in atomic units $\hbar^2 = \mu = Z = e = 1$,

$$A_c = (n+1+\eta)^2 - \eta(\eta+1) + \frac{(D+2l-2)^2}{4} - \frac{1}{4}. \quad (67)$$

Let us now find the corresponding radial part of the wave function. Using $\sigma(z)$ and $\pi(z)$ in Eqs (57) and (61), we obtain

$$\phi(z) = z^\varepsilon (1-z)^{(\eta+1)/2}, \quad (68)$$

$$\rho(z) = z^{2\varepsilon} (1-z)^{2\eta+1}, \quad (69)$$

$$y_{nl}(z) = C_n z^{-2\varepsilon} (1-z)^{-(2\eta+1)} \frac{d^n}{dz^n} \left[z^{n+2\varepsilon} (1-z)^{n+2\eta+1} \right]. \quad (70)$$

The functions $y_{nl}(z)$ are, up to a numerical factor, are in the form of Jacobi polynomials, i.e., $y_{nl}(z) \simeq P_n^{(2\varepsilon, 2\eta+1)}(1-2z)$, valid physically in the interval $(0 \leq r < \infty \rightarrow 0 \leq z \leq 1)$ [52]. Therefore, the radial part of the wave functions can be found by substituting Eqs. (68) and (70) into $R_{nl}(z) = \phi(z)y_{nl}(z)$ as

$$g_{nl}(z) = N_{nl} z^\varepsilon (1-z)^{1+\eta} P_n^{(2\varepsilon, 2\eta+1)}(1-2z), \quad (71)$$

where ε and Λ are given in Eq. (65) and N_{nl} is a normalization constant. This equation satisfies the requirements; $R_{nl}(z) = 0$ as $z = 0$ ($r \rightarrow \infty$) and $R_{nl}(z) = 0$ as $z = 1$ ($r = 0$). Therefore, the wave functions, $g_{nl}(z)$ in Eq. (71) is valid physically in the closed interval $z \in [0, 1]$ or $r \in (0, \infty)$. Further, the wave functions satisfy the normalization condition

$$\int_0^\infty |g_{nl}(r)|^2 dr = 1 = b \int_0^1 z^{-1} |g_{nl}(z)|^2 dz, \quad (72)$$

where N_{nl} can be determined via

$$1 = b N_{nl}^2 \int_0^1 z^{2\varepsilon-1} (1-z)^{2\eta+2} \left[P_n^{(2\varepsilon, 2\eta+1)}(1-2z) \right]^2 dz. \quad (73)$$

Following Ref. [36], we find the normalization constant

$$N_{nl} = \frac{1}{\sqrt{s(n)}}, \quad (74)$$

where

$$s(n) = b(-1)^n \frac{\Gamma(n+2\eta+2)\Gamma(n+2\varepsilon+1)^2}{\Gamma(n+2\varepsilon+2\eta+2)} \times \sum_{p,r=0}^n \frac{(-1)^{p+r}\Gamma(n+2\varepsilon+r-p+1)(p+2\eta+2)}{p!r!(n-p)!(n-r)!\Gamma(n+2\varepsilon-p+1)\Gamma(2\varepsilon+r+1)(n+2\varepsilon+r+2\eta+2)}. \quad (75)$$

Therefore, we may express the normalized total wave functions as

$$\psi(\mathbf{x}) = N_{nl} r^{-(D-3)/2} e^{-\varepsilon r/b} (1 - e^{-r/b})^{1+\eta} P_n^{(2\varepsilon, 2\eta+1)}(1 - 2e^{-r/b}) \exp(\pm i l_1 \theta_1) \\ \times N_{n_{D-1}} \sin(\theta_{D-1})^{l_{D-2}} P_{n_{D-1}}^{(\tilde{\Lambda}_{D-2}, \tilde{\Lambda}_{D-2})}(\cos \theta_{D-1}) \cdot \prod_{j=2}^{D-2} N_{n_j} (\sin \theta_j)^{l_{j-1}} P_{n_j}^{(\tilde{\Lambda}_{j-1}, \tilde{\Lambda}_{j-1})}(\cos \theta_j) \quad (76)$$

To show the accuracy of our results, we calculate the energy eigenvalues for various n and l quantum numbers with three different values of the parameters α . As seen in Table 1, the energy eigenvalues for different quantum numbers are obtained numerically for $D = 2$ and $D = 4$ cases.

From Eq. (66), we have seen that two interdimensional states are degenerate whenever [53]

$$(n, l, D) \rightarrow (n, l \pm 1, D \mp 2). \quad (77)$$

Thus, a knowledge of $E_{nl}^{(D)}$ for $D = 2$ and $D = 3$ provides the information necessary to find $E_{nl}^{(D)}$ for other higher dimensions.

For example, $E_{0,4}^{(2)} = E_{0,3}^{(4)} = E_{0,2}^{(6)} = E_{0,1}^{(8)}$. This is the same transformational invariance described for bound states of free atoms and molecules [54,55] and demonstrates the existence of interdimensional degeneracies among states of the confined Manning-Rosen potential.

As an example of incidental degeneracy, Table 1 presents the results of the confined $2D$ and $4D$ Manning-Rosen energies at several radii of confinement for various n and l states.

IV. DISCUSSIONS

In this work, we have utilized NU method to solve the D -dimensional SE for the Manning-Rosen model potential with the angular momentum $l \neq 0$ states. We have derived the binding energy spectra in Eq. (66) and their corresponding wave functions in Eq. (71).

Let us study special cases. We have shown that for $\alpha = 0$ (1), the present solution reduces to the one of the Hulthén potential [16,18,19]:

$$V^{(H)}(r) = -V_0 \frac{e^{-\delta r}}{1 - e^{-\delta r}}, \quad V_0 = Ze^2\delta, \quad \delta = b^{-1} \quad (78)$$

where Ze^2 is the strength and δ is the screening parameter and b is the range of potential. If the potential is used for atoms, the Z is identified with the atomic number. This can be achieved by setting $\eta = \frac{1}{2}(D + 2l - 3)$, hence, the energy for $l \neq 0$ states

$$E_{nl} = -\frac{[4A - (2n + D + 2l - 1)^2]^2 \hbar^2}{32\mu b^2(2n + D + 2l - 1)^2}, \quad 0 \leq n, l < \infty. \quad (79)$$

and for s -wave ($l = 0$) states

$$E_n = -\frac{[A - (n + 1)^2]^2 \hbar^2}{8\mu b^2(n + 1)^2}, \quad 0 \leq n < \infty \quad (80)$$

Essentially, these results coincide with those obtained by the Feynman integral method [31] and the standard way [32,33], respectively. Furthermore, if taking $b = 1/\delta$ and identifying $\frac{A\hbar^2}{2\mu b^2}$ as $Ze^2\delta$, we are able to obtain

$$E_{nl} = -\frac{\mu(Ze^2)^2}{\hbar^2} \left[\frac{1}{(2n+D+2l-1)} - \frac{\hbar^2\delta}{8Ze^2\mu}(2n+D+2l-1) \right]^2, \quad (81)$$

which coincides with those of Refs. [16,18]. With natural units $\hbar^2 = \mu = Z = e = 1$, we have

$$E_{nl} = - \left[\frac{1}{(2n+D+2l-1)} - \frac{(2n+D+2l-1)}{8} \delta \right]^2, \quad (82)$$

which coincides with Refs. [16,33].

The corresponding radial wave functions are expressed as

$$R_{nl}(r) = N_{nl} e^{-\delta\epsilon r} (1 - e^{-\delta r})^{(D+2l-1)/2} P_n^{(2\epsilon, D+2l-2)}(1 - 2e^{-\delta r}) r^{-(D-3)/2}, \quad (83)$$

where

$$\epsilon = \frac{2\mu Ze^2}{\hbar^2\delta} \left[\frac{1}{(2n+D+2l-1)} - \frac{\hbar^2\delta}{8Ze^2\mu}(2n+D+2l-1) \right], \quad 0 \leq n, l < \infty, \quad (84)$$

which coincides for the ground state with that given in Eq. (6) by Gönül *et al* [18]. In addition, for $\delta r \ll 1$ (i.e., $r/b \ll 1$), the Hulthén potential turns to become a Coulomb potential: $V(r) = -Ze^2/r$ with energy levels and wavefunctions:

$$E_{nl} = -\frac{4\epsilon_0}{(2n+D+2l-1)^2}, \quad n = 0, 1, 2, \dots$$

$$\epsilon_0 = \frac{Z^2\hbar^2}{2\mu a_0^2}, \quad a_0 = \frac{\hbar^2}{\mu e^2} \quad (85)$$

where $\epsilon_0 = 13.6 \text{ eV}$ and a_0 is Bohr radius for the Hydrogen atom. The wave functions are

$$R_{nl} = N_{nl} \exp \left[-\frac{8\mu Ze^2}{\hbar^2} \frac{r}{(2n+D+2l-1)} \right] r^{(D+2l-1)/2} P_n^{\left(\frac{2\mu Ze^2}{\hbar^2\delta(2n+D+2l-1)}, D+2l-2 \right)}(1 + 2\delta r)$$

which coincide with Refs. [3,16,22].

V. COCLUDING REMARKS

In this work, we have extended the approximate solutions of the l -wave Schrödinger equation with the Manning-Rosen potential to D -dimensions. The special cases for $\alpha = 0, 1$

are discussed. The results are found to be in good agreement with those obtained by other methods in $3D$ for short potential range, small α and l [36]. These numerical solutions have also extended to various dimensional space, $D = 2$ and $D = 4$ systems. We have also studied two special cases for $l = 0$, $l \neq 0$ and Hulthén potential. The results we have ended up show that the NU method constitute a reliable alternative way in solving the exponential potentials.

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TABLES

TABLE I. Eigenvalues for $2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f$ and $6g$ states in atomic units ($\hbar = \mu = 1$) and for $\alpha = 0.75$ and $\alpha = 1.5$, $A = 2b$.

states	$1/b$	$D = 2^a$			$D = 4^b$		
		$\alpha = 0.75$	$\alpha = 0, 1$	$\alpha = 1.5$	$\alpha = 0.75$	$\alpha = 0, 1$	$\alpha = 1.5$
$2p$	0.025	-0.241087728	-0.209898003	-0.140949065	-0.070734690	-0.067988281	-0.058898861
	0.050	-0.227946676	-0.197925347	-0.131737328	-0.059344084	-0.056953125	-0.049054156
	0.075	-0.215173874	-0.186304253	-0.122836866	-0.048952839	-0.046894531	-0.040109106
	0.100	-0.202769319	-0.175034722	-0.114247678	-0.039560954	-0.037812500	-0.032063712
$3p$	0.025	-0.074279113	-0.067988281	-0.051933432	-0.030209821	-0.029273358	-0.026068346
	0.050	-0.062813564	-0.056953125	-0.042142549	-0.020395577	-0.019644452	-0.017092049
	0.075	-0.052308602	-0.046894531	-0.033373420	-0.012502916	-0.011929608	-0.010003237
	0.100	-0.042764227	-0.037812500	-0.025626042	-0.006531840	-0.006128827	-0.004801908
$3d$	0.025	-0.070734690	-0.067988281	-0.058898861	-0.029833656	-0.029273358	-0.027228277
	0.050	-0.059344084	-0.056953125	-0.049054156	-0.020047209	-0.019644452	-0.018176769
	0.075	-0.048952839	-0.046894531	-0.040109106	-0.012199670	-0.011929608	-0.010947973
$4p$	0.025	-0.031448122	-0.029273358	-0.023381941	-0.014180352	-0.013773389	-0.012357598
	0.050	-0.021545731	-0.019644452	-0.014606136	-0.006296995	-0.006019483	-0.005072360
	0.075	-0.013510134	-0.011929608	-0.007885467	-0.001570215	-0.001429639	-0.000978205
$4d$	0.025	-0.030209821	-0.029273358	-0.026068346	-0.014011823	-0.013773389	-0.012892982
	0.050	-0.020395577	-0.019644452	-0.017092049	-0.006162813	-0.006019483	-0.005494347
	0.075	-0.012502916	-0.011929608	-0.010003237	-0.001492711	-0.001429639	-0.001204122
$4f$	0.025	-0.029833656	-0.029273358	-0.027228277	-0.013929374	-0.013773389	-0.013182139
	0.050	-0.020047209	-0.019644452	-0.018176769	-0.006097355	-0.006019483	-0.005724889
	0.075	-0.012199670	-0.011929608	-0.010947973	-0.001455297	-0.001429639	-0.001333163
$5p$	0.025	-0.014732070	-0.013773389	-0.011100961	-0.007127957	-0.006916484	-0.006175251
$5d$	0.025	-0.014180352	-0.013773389	-0.012357598	-0.006506751	-0.006392207	-0.005967020

$5f$	0.025	-0.014011823	-0.013773389	-0.012892982	-0.006465489	-0.006392207	-0.006113207
$5g$	0.025	-0.013929374	-0.013773389	-0.013182139	-0.006440958	-0.006392207	-0.006204004
$6p$	0.025	-0.006866319	-0.006392207	-0.005056211	-0.002734814	-0.002635101	-0.002286461
$6d$	0.025	-0.005435481	-0.006392207	-0.005695750	-0.002691847	-0.002635101	-0.002424502
$6f$	0.025	-0.006506751	-0.006392207	-0.005967020	-0.002670817	-0.002635101	-0.002499036
$6g$	0.025	-0.006465489	-0.006392207	-0.006113207	-0.002658317	-0.002635101	-0.002545374

^aTwo-dimensional Schrodinger equation.

^bFour-dimensional Schrodinger equation.